

TRANSLATIONAL INVARIANCE AND NONCOMMUTATIVE FIELD THEORIES

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Implications of noncommutative field theories with commutator of the coordinates of the form $[x^\mu, x^\nu] = i \Lambda^{\mu\nu}_\omega x^\omega$ with nilpotent structure constants are investigated. It is shown that a free quantum field theory is not affected by noncommutativity, but that invariance under translations is broken and hence the energy-momentum conservation is not respected. The new energy-momentum law is expressed by a Poincaré-invariant equation and the resulting kinematics is developed and applied to the astrophysical puzzle related with the observed violation of the GZK cutoff.

Dedicated to the memory of Luís Guisado

1 Introduction

Achieving a consistent theory of quantum gravity is one of the major goals of XXI century physics. String/M-theory is the best candidate so far for this synthesis:

1. It is a finite quantum theory for gravity and contains, thanks to the mechanism of cancellation of gauge and gravitational anomalies, grand unified theories in a constrained way.
2. It is naturally supersymmetric, and hence leads to models that are free from the hierarchy problem.
3. It has black hole solutions, and therefore allows addressing key issues associated with those singular objects, such as the physics underlying their entropy and their presumed non-unitary evolution.
4. It is a natural framework for many important ideas and techniques in the field theory such as Supergravity, Kaluza-Klein mechanism, conformal field theory, non-commutative geometry, braneworld scenarios, etc.

Despite these appealing features, string theory has so far neither provided a decisive insight toward a solution of the cosmological constant problem¹ nor has unambiguously suggested a clear cut set of phenomenological signatures. This last task would be simpler if among the predictions of string theory were the breaking of fundamental symmetries such as Lorentz invariance, CPT symmetry and the Equivalence Principle in phenomenologically testable ranges. It is remarkable that the spontaneous breaking of Lorentz and CPT symmetries can occur in string field theory^{2,3}, but phenomenological implications are shown to be fairly subtle, probably suppressed by powers of the Planck mass. The same can be stated about the Equivalence Principle⁴.

Given these difficulties, it is particularly relevant to broaden the scope of the

search for experimental evidences of quantum gravity phenomena. Even though it is not beyond dispute, it has been argued that evidence for the breaking of Lorentz symmetry may have already been encountered in cosmic ray physics and high-energy astrophysics:

1. In the observation of ultra-high energy cosmic rays^{5,6,7,8} with energies beyond the Greisen-Zatsepin-Kuzmin (GZK) cutoff, $E_{GZK} \simeq 4 \times 10^{19} \text{ eV}$ ⁹, even though, the compatibility of these observations with the preliminary measurements of the energy spectrum by the HiRes Collaboration is still under debate¹⁰. These events, are a challenge to present knowledge to accelerate cosmic particles and may require the violation of Lorentz invariance as an explanation whether their sources are shown to lie beyond $50 - 100 \text{ Mpc}$ ¹¹. This arises as breaking of Lorentz symmetry suppresses resonant scattering reactions of the primaries with photons of the Cosmic Microwave Background (CMB)^{12,13,14,15}. We mention that no conclusive correlation with astrophysical sources has ever been found¹⁶.
2. In the observation of gamma radiation from distant sources such as Markarian 421 and Markarian 501 blazars with energies above 20 TeV ¹⁷. These observations suggest a violation of Lorentz symmetry as otherwise there should exist a strong attenuation of fluxes beyond 100 Mpc of γ -rays with energies higher than 10 TeV by the diffuse extragalactic background of infrared photons due to pair creation¹⁸.
3. In the evolution of air showers produced by ultra high-energy hadronic particles which suggests that pions are more stable than predicted¹⁹.

As already mentioned, Lorentz invariance can be spontaneously broken due to non-trivial solutions in string field theory², but may also arise in loop quantum gravity^{20,21}, and in quantum gravity inspired spacetime foam scenarios²². A violation of the Lorentz symmetry may also lead to the breaking of CPT symmetry³. There exists a workable extension of the Standard Model inspired in string field theory that incorporates violations of Lorentz and CPT symmetries²³. In the context of this extension several questions can be addressed, such as the violation of GZK cutoff¹⁵, the generation of the baryon asymmetry of the Universe²⁴ and of primordial magnetic fields²⁵. The breaking of Lorentz symmetry also arises in models where the electromagnetic coupling evolves^{26,27} and may also be related to the vacuum energy density²⁸. Several other consequences to particle physics have already been worked out²⁹.

Another important class of theories where Lorentz invariance is not respected are noncommutative field theories³⁰. Noncommutative theories have been extensively studied given that they naturally emerge in string theory³¹, and also due to their interesting properties and implications for field theory³². In what concerns the coupling with gravity in noncommutative theories, a noncommutative scalar field theory has already been examined³³.

In what follows we shall consider classical and quantum field theory features of models where the noncommutativity of the coordinates has the following form

$$[x^\mu, x^\nu] = i \Lambda^{\mu\nu}{}_\omega x^\omega, \quad (1)$$

together with the condition of nil-potency specified below. This leads to a violation of the symmetry under translations and, consequently, requires a reformulation of the energy-momentum conservation³⁴. This work has been developed in collaboration with Luís Guisado, who tragically died in a car crash on June 28th, 2003. Luís was a bright 23 years old graduate student and was regarded as one of the great hopes of portuguese theoretical physics. I would like to dedicate this contribution to the memory of his kind and friendly person.

2 Mathematical Considerations

A noncommutative and associative product can be defined through the Lie-algebra commutator Eq. (1), where $\Lambda^{\mu\nu}{}_\omega$ is a real tensor with units of mass⁻¹ and $\Lambda^{\mu\nu}{}_\omega = -\Lambda^{\nu\mu}{}_\omega$. Associativity implies in the Jacobi identity

$$\Lambda^{\mu\nu}{}_\omega \Lambda^{\omega\alpha}{}_\beta + \Lambda^{\nu\alpha}{}_\omega \Lambda^{\omega\mu}{}_\beta + \Lambda^{\alpha\mu}{}_\omega \Lambda^{\omega\nu}{}_\beta = 0. \quad (2)$$

A noncommutative Fourier mode can be defined by

$$e_*^{ik \cdot x} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \overbrace{(k \cdot x) * \dots * (k \cdot x)}^{n \text{ factors}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} (k \cdot x)_*^n, \quad (3)$$

so that the functional space spanned by these Fourier modes, with elements of the form

$$f(x) = \int \frac{d^n k}{(2\pi)^n} \tilde{f}(k) e_*^{ik \cdot x} \quad (4)$$

reduces, in the commutative limit, to the usual Hilbert space.

The product of two generic functions is given by

$$f * g = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \tilde{f}(k) \tilde{g}(q) e_*^{ik \cdot x} * e_*^{iq \cdot x}, \quad (5)$$

where the functions are expressed in terms of their noncommutative Fourier expansion. This product is completely determined if the product of two Fourier modes $e_*^{ik \cdot x} * e_*^{iq \cdot x}$ can be evaluated. This is achieved through the Baker-Hausdorff formula

$$e_*^{ik \cdot x} * e_*^{iq \cdot x} = \exp_* \left\{ i(k+q) \cdot x + \frac{1}{2} [ik \cdot x, iq \cdot x] + \dots \right\}, \quad (6)$$

where the dots stand for higher order commutators. Since the commutators obey

$$[x^{\mu_1}, [x^{\mu_2}, \dots, [x^{\mu_n}, x^\nu] \dots]] \propto i^n x^\omega, \quad (7)$$

the product of two Fourier modes is a Fourier mode

$$e_*^{ik \cdot x} * e_*^{iq \cdot x} = e_*^{i[k+q+V(k,q)] \cdot x} \quad (8)$$

with V being determined by the Baker-Hausdorff expansion:

$$V_\omega(k, q) = k_\mu q_\nu \Lambda^{\mu\nu}{}_\lambda \left[-\frac{1}{2} \delta_\omega^\lambda + \frac{k_\alpha - q_\alpha}{12} \Lambda^{\alpha\lambda}{}_\omega \right] + O(\Lambda^3). \quad (9)$$

2.1 Quadratic Actions

In order to construct actions, a star-integration must be defined. In the functional space whose elements are of the form Eq. (4), any function can be integrated once the integral of a Fourier mode is known. Hence, in the following the star-integration is introduced

$$\int_* d^n x e^{ir \cdot x} = (2\pi)^n \delta(r), \quad (10)$$

which yields the usual integration in the commutative limit.

Consider now the star-integral

$$I = \int_* d^n x f * g \quad (11)$$

which, in Fourier space, is written as

$$I = \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \tilde{f}(k) \tilde{g}(q-k) \int_* d^n x e^{i[q+V(k,q-k)] \cdot x}, \quad (12)$$

following that

$$I = \int \frac{d^n k}{(2\pi)^n} d^n q \tilde{f}(k) \tilde{g}(q-k) \delta(q + V(k, q-k)). \quad (13)$$

Taking the structure constants nilpotent, that is, for $n > n_*$

$$\Lambda^{\mu_1 \nu}{}_{\omega_1} \Lambda^{\mu_2 \omega_1}{}_{\omega_2} \dots \Lambda^{\mu_n \omega_{n-1}}{}_{\omega_n} = 0, \quad (14)$$

then

$$\delta(q + V(k, q-k)) = \frac{\delta(q)}{\left| \det \left(\delta_\nu^\mu - \frac{\partial V_\nu}{\partial q_\mu} \right) \right|} = \delta(q) \quad (15)$$

since $\det(1 + M) = 1$ if $M^n = 0$, which holds if Λ is nilpotent. Therefore

$$I = \int \frac{d^n k}{(2\pi)^n} \tilde{f}(k) \tilde{g}(-k) = \int d^n x f_C(x) g_C(x) \quad (16)$$

where f_C, g_C are inverse Fourier transforms using commutative Fourier modes

$$f_C(x) = \int \frac{d^n k}{(2\pi)^n} \tilde{f}(k) e^{ik \cdot x}. \quad (17)$$

Equation (16) states that, in momentum space, quadratic terms in the Lagrangian are the same as their commutative counterparts. In particular, this implies that free propagators remain unchanged.

3 Violation of Momentum Conservation

From above considerations one can conclude that the quadratic part of a Lagrangian is not changed and, hence, the free theory is the same as the commutative one. Thus, the free Green function is equal to the commutative case and the dispersion relation $\epsilon^2 = p^2 + m^2$ is unchanged, since it is given by the poles of the free propagator. However, one finds that interactions are altered by noncommutativity.

Consider a noncommutative field theory, with generic fields A_i and an interaction term

$$S_I = \int_* d^n x M_{i_1 \dots i_m} A_{i_1} * \dots * A_{i_m}, \quad (18)$$

where $M_{i_1 \dots i_m}$ are constants.

In the momentum space one obtains

$$S_I = \int \left[\prod_{i=1}^m \frac{d^n k_i}{(2\pi)^n} \right] \tilde{M}_{i_1 \dots i_m}(\underline{k}_m) \tilde{A}_{i_1}(k_1) \dots \tilde{A}_{i_m}(k_m), \quad (19)$$

where the notation $\underline{k}_m = (k_1, \dots, k_m)$ has been used. The interaction in momentum space is given by

$$\tilde{M}_{i_1 \dots i_m}(\underline{k}_m) = M_{i_1 \dots i_m} \int_* d^n x e^{ik_1 \cdot x} * \dots * e^{ik_m \cdot x}. \quad (20)$$

Notice that in Eq. (19) the variables k_i are mute, so one must sum over all π permutations of the indices i_m :

$$S_I = \int \left[\prod_{i=1}^m \frac{d^n k_i}{(2\pi)^n} \right] \tilde{M}_{i_1 \dots i_m}^{symm}(\underline{k}_m) \tilde{A}_{i_1}(k_1) \dots \tilde{A}_{i_m}(k_m), \quad (21)$$

where

$$\tilde{M}_{i_1 \dots i_m}^{symm}(\underline{k}_m) = \frac{1}{m!} \sum_{\pi \text{ perm.}} (-)^{N(\pi)} \tilde{M}_{i_{\pi(1)} \dots i_{\pi(m)}}(\underline{k}_{\pi(m)}). \quad (22)$$

In order to evaluate Eq. (20), one uses

$$e_*^{ik_1 \cdot x} * \dots * e_*^{ik_m \cdot x} = \exp_* \left\{ i \sum_{j=1}^m k_j \cdot x + i V^m(\underline{k}_m) \cdot x \right\} \quad (23)$$

where

$$V^m(\underline{k}_m) = V^{m-1}(\underline{k}_{m-1}) + V \left(\sum_{i=1}^{m-1} k_i + V^{m-1}(\underline{k}_{m-1}), k_m \right) \quad (24)$$

with $V^2(\underline{k}_2) = V(k_1, k_2)$. This yields both the noncommutative energy-momentum law and the noncommutative vertex

$$\tilde{M}_{i_1 \dots i_m}(\underline{k}_m) = (2\pi)^n \delta \left(\sum_{i=1}^m k_i + V^m(\underline{k}_m) \right) M_{i_1 \dots i_m}. \quad (25)$$

Hence, the noncommutative energy-momentum law for the full theory $\tilde{M}_{i_1 \dots i_m}^{symm}$ vertex reads

$$\sum_{i=1}^m k_i + V^m(\underline{k}_{\pi(m)}) = 0 \quad (26)$$

for all $m!$ permutations of indices, π .

Thus, it can be seen that the energy-momentum conservation is violated as the theory is not invariant under translations. Indeed, in a translation $x^\mu \rightarrow x^\mu + b^\mu$, the commutator of the coordinates is changed by

$$[x^\mu, x^\nu] \rightarrow i \Lambda^{\mu\nu}{}_\omega x^\omega + i \theta^{\mu\nu}, \quad (27)$$

that is, a constant term $\theta^{\mu\nu} = \Lambda^{\mu\nu}{}_\omega b^\omega$ is added to the commutator of the coordinates. So, the interaction vertex becomes

$$\tilde{M}_{i_1 \dots i_m}(\underline{k}_m) \rightarrow (2\pi)^n \delta \left(\sum_{i=1}^m k_i + V^m(\underline{k}_{\pi(m)}) \right) M_{i_1 \dots i_m} \exp \{ i \theta^m(\underline{k}_m) \} \quad (28)$$

where

$$\theta^m(\underline{k}_m) = \theta^{m-1}(\underline{k}_{m-1}) + \theta \left(\sum_{i=1}^{m-1} k_i + V^{m-1}(\underline{k}_{m-1}), k_m \right) \quad (29)$$

and $\theta^2(\underline{k}_2) = k_{1\mu} \theta^{\mu\nu} k_{2\nu}$. Therefore, the interaction vertex is changed by an overall oscillating momentum-dependent factor which breaks invariance under translations. This shows that translations give always rise to a constant term in the noncommutative tensor.

4 Kinematic Applications

4.1 Preliminaries

The first non-trivial consequence arising from the new interaction vertex takes place when considering three particles. The energy-momentum equation is found to be

$$k_1 + k_2 + k_3 + V(k_1, k_2) + V(k_1 + k_2, k_3) = 0 \quad (30)$$

and similar expressions for all permutations of the indices.

Let us first consider the model

$$\Lambda^{\mu_1 \nu}_{\omega_1} \Lambda^{\mu_2 \omega_1}_{\omega_2} = 0 \quad , \quad (31)$$

which is consistent with the Jacobi identity Eq. (2).

The energy-momentum equation reads

$$k_1 + k_2 + k_3 + V(k_1, k_2) = 0 \quad , \quad (32)$$

where

$$V_\omega(k_1, k_2) = \frac{1}{2} k_{1\mu} k_{2\nu} \Lambda^{\mu\nu}_\omega \quad . \quad (33)$$

Eq. (31) admits non-trivial covariant solutions. For instance, consider a constant antisymmetric tensor $\Lambda^{\mu\nu} = -\Lambda^{\nu\mu}$ with non-trivial kernel, $\det \Lambda = 0$, and a non-vanishing vector r^ω belonging to this kernel. Hence a solution is given by

$$\Lambda^{\mu\nu\omega} = \Lambda^{\mu\nu} r^\omega \quad . \quad (34)$$

In four dimensions one can parametrize $\Lambda^{\mu\nu}$ with two spatial vectors \vec{E} and \vec{B}

$$\Lambda^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad , \quad r_\nu = \begin{pmatrix} r_0 \\ r_x \\ r_y \\ r_z \end{pmatrix} \quad . \quad (35)$$

Condition Eq. (34) implies that

$$r^2 = |\vec{r}|^2 \left[\left(\frac{B}{E} \sin \delta \right)^2 - 1 \right] \quad , \quad (36)$$

with δ being the angle between \vec{B} and \vec{r} . The massless, massive and tachyon regimes of V can be easily identified. Assuming that $\Lambda^{\mu\nu}$ is a Lorentz tensor, there are always inertial frames where \vec{E} is non-vanishing, and the above expression holds only for such frames. If $B < E$ (a Lorentz-invariant inequality) then r^ω behaves like a tachyon; otherwise, the behaviour of r^ω will depend on δ .

From the momentum expression, Eq. (32), one gets

$$\Lambda^{\mu\nu} (k_1 + k_2 + k_3)_\nu = 0 \quad , \quad (37)$$

which states that the vector sum of the momenta belongs to the non-trivial kernel of the noncommutative tensor. There follows the expressions

$$\Lambda^{\mu\nu} k_\nu = \begin{pmatrix} \vec{E} \cdot \vec{k} \\ -k_0 \vec{E} + \vec{B} \times \vec{k} \end{pmatrix} \quad (38)$$

and

$$q_\mu \Lambda^{\mu\nu} k_\nu = \vec{E} \cdot (q_0 \vec{k} - k_0 \vec{q}) + \vec{B} \cdot (\vec{k} \times \vec{q}) . \quad (39)$$

Eqs. (37) and (38) imply that the three-momentum is conserved along the direction of \vec{E} . Energy is conserved if the total three-momentum $\sum \vec{k}_i$ lies along the direction of \vec{B} . Also,

$$k_1 \Lambda k_2 = -k_1 \Lambda k_3 = k_2 \Lambda k_3 \quad (40)$$

and one is required only to study Eq. (32) with $V(k_1, k_2)$ and $-V(k_1, k_2)$. Notice that the second case is obtained by performing the transformation $\vec{E}, \vec{B} \rightarrow -\vec{E}, -\vec{B}$.

4.2 The GZK cutoff

Let us discuss the GZK cutoff in the context of our noncommutative model, considering the dominant resonance

$$p + \gamma_{CMB} \rightarrow \Delta_{1232} . \quad (41)$$

A violation of the GZK cutoff can arise in the context of the model

$$\Lambda^{\mu_1 \nu}_{\omega_1} \Lambda^{\mu_2 \omega_1}_{\omega_2} \Lambda^{\mu_3 \omega_2}_{\omega_3} = 0 , \quad (42)$$

with $\Lambda^{\mu_1 \nu}_{\omega_1} \Lambda^{\mu_2 \omega_1}_{\omega_2} \neq 0$. This cannot be implemented by model Eq. (34), as the analysis is fairly complicate. The equation for the momentum is given by

$$(k_1 + k_2 + k_3)_\omega = k_{1\mu} k_{2\nu} \Lambda^{\mu\nu}_\lambda \left[-\frac{1}{2} \delta_\omega^\lambda + \frac{(k_1 - k_2)_\alpha}{12} \Lambda^{\alpha\lambda}_\omega \right] \quad (43)$$

where Eq. (30) has been recursively used as well as the fact that the cubic terms in Λ vanish.

Condition Eq. (43) can be replaced by a simpler one, more suitable for phenomenological considerations. Dropping quadratic terms in the momentum and taking into account that the proton has the highest energy and the Δ the second highest energy, one can write the new momentum equation for the reaction (41) as

$$(k_p + k_\gamma)^\mu = k_\Delta^\mu - s^\mu \frac{\epsilon_p^2}{M^2} \epsilon_\Delta , \quad (44)$$

where the dimensionless vector s^μ is of the order of unity and M is the typical noncommutative mass scale. In this case, the process is impossible if $s^0 > 0$ and $\epsilon_p > M$, which sets the scale of noncommutativity.

It is not difficult to see that a cubic term in the dispersion equation^{35,36,37} can explain the violation of this cutoff, even though general arguments, based on coordinate invariance and causality, may prevent this term if 4-momentum is conserved³⁸. These objections do not apply to our proposal given the breaking of translational invariance. In fact, Eq. (44) can be obtained postulating a new equation of dispersion by the substitution

$$k^\mu \rightarrow k^\mu + s^\mu \frac{\epsilon^2}{M^2} \lambda \quad (45)$$

where λ represents the typical energy of the product of the reaction. This leads to the following dispersion relation

$$m^2 = \epsilon^2 - p^2 + 2s^\mu v_\mu \frac{\lambda}{M^2} \epsilon^3, \quad (46)$$

where $v^\mu = (1, \vec{v})$ is the four-vector velocity, which is assumed to be ultra relativistic. Only the lower order terms of the correction were kept. Thus, it is as if a cubic term had been introduced into the dispersion relation and the GZK cutoff is evaded if $M \simeq 4 \times 10^{19} eV$.

5 Conclusions

In this work we have discussed a noncommutative field theory where the coordinates have a Lie-algebra commutator as Eq. (1) with nilpotent structure constants. This breaks Lorentz symmetry as well as translational invariance. Free theory is unchanged so the propagators and the dispersion relations are not modified. Interaction however, lead to a new energy-momentum law, which follows from the breaking of translational invariance. The kinematics of such law was established and considered as a possible explanation for the violation of the GZK cutoff, if one chooses the noncommutative mass scale at $M \simeq 4 \times 10^{19} eV$. The use of the present results to the other astrophysical puzzles will be considered elsewhere.

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